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## Abstract:

In this paper, We propose a novel bivariate probability model constructed by coupling exponentiated sine exponential marginal distributions via the Farlie-Gumbel-Morgenstern (FGM) copula, termed the FGM Bivariate Exponentiated Sine Exponential (FGMBESE) distribution. Key statistical properties of the model are derived and examined. The parameters of the proposed distribution are estimated using the method of maximum likelihood. Finally, the flexibility and effectiveness of the proposed model are illustrated through real data analysis.

**Keywords**: FGM Copula, Bivariate modeling, Maximum Likelihood Estimation (MLE), Reliability analysis, Lifetime data.

## 1. Introduction

The classical probability distributions plays a significant role in the various fields of applied sciences such as reliability, economics, medical sciences, and other related areas. Among these distributions, the exponential distribution has been widely used for modeling lifetime data due to its mathematical simplicity. However, real-life data often exhibit more complex behaviors that cannot be adequately captured by classical models. This limitation has motivated researchers to propose various generalizations and extensions of classical distributions in order to achieve greater flexibility and improved data fitting. (see Johnson et al., (1995), Sarhan and Kundu, (2009), Sen et al., (2016)).

In recent years, lifetime data arise in pairs and exhibit dependence between the variables. Consequently, bivariate lifetime models have gained considerable attention in statistical literature. Several approaches have been developed to construct bivariate distributions based on well-known univariate distributions such as exponential, Weibull, Pareto, gamma, and log-normal distribution (see, for example, Gumbel (1960), Marshall and Olkin (1967), Sankaran and Nair (1993), Kundu and Gupta (2009), Sarhan et al, (2011)). These models are particularly important when the assumption of independence between variables is unrealistic.

Copula functions provide a powerful framework for constructing bivariate distributions with flexible dependence structures while preserving specified marginal distributions. According to Sklar's theorem (Sklar, 1973), any multivariate distribution can be expressed in terms of its marginals and a copula function. Among various copula families, the Farlie–Gumbel–Morgenstern (FGM) copula is attractive due to its simple form and ability to model both positive and negative dependence.

Recently, the exponentiated sine-G (ES-G) family has been introduced as a flexible class of lifetime distributions capable of modeling various hazard rate shapes. Motivated by its flexibility and analytical tractability, this paper proposes a new bivariate model by combining exponentiated sine exponential marginals with the FGM copula, referred to as the FGMBESE distribution.

The main objective of this article is to study the statistical properties of the FGMBESE distribution and to estimate its parameters using both maximum likelihood and Bayesian methods. Closed-form expressions for the distribution function, product moments, moment generating function, and hazard rate are derived. A real data application is also presented to illustrate the usefulness of the proposed model.

The paper is structured as follows: In section 2, the Copula Based Construction is introduced. Section 3 discusses the FGM bivarite Exponentaited sine Exponential distribution. In section 4, discusses its statistical properties. Some measures of dependence based on copulas FGM are discussed in section 5. Estimation using maximum likelihood method is addressed in section 6. An real data application is provided in section 7, respectively. Finally, concluding remarks are addressed in section 8.

## 2. Model Construction

In this section, the construction of the proposed bivariate model is presented. First, the general framework of copula-based modeling is briefly reviewed. Then, the Farlie–Gumbel–Morgenstern (FGM) copula is introduced as the dependence structure used to build the bivariate distribution.

### 2.1 Copula Foundations

Copulas play a fundamental role in the construction of multivariate distributions by linking marginal distributions to form a joint distribution with a specified dependence structure. The theoretical foundation of copula-based modeling is provided by Sklar's theorem (Sklar, 1973), which states that for any bivariate random variables X and Y with continuous marginal distribution functions $F_1(x;\zeta_1)$and $F_2(y;\zeta_2)$, there exists a unique copula function C(.,.) such that the joint cumulative distribution function (CDF) can be written as

$$F_{X,Y}(x,y) = C(F_1(x;\zeta_1),F_2(y;\zeta_2))$$

If the copula function is differentiable, the corresponding joint probability density function (PDF) is given by

$$f_{X,Y}(x,y) = f_1(x;\zeta_1)f_2(y;\zeta_2)c(F_1(x;\zeta_1),F_2(y;\zeta_2))$$

Where $f_1(.)f_2(.)$ are the marginal PDFs and c(.,.) denotes the copula density function. This representation allows the dependence structure to be modeled independently of the marginal distributions, providing great flexibility in bivariate modeling.

## 2.2 FGM Copula

Among the various copula families, the Farlie–Gumbel–Morgenstern (FGM) copula is one of the simplest and most widely used dependence models. Originally proposed by Morgenstern and later generalized by Farlie and Gumbel (Gumbel, 1960), the FGM copula is characterized by its analytical simplicity and computational efficiency.

The joint cdf and joint pdf for FGM copula given as following respectively

$$C_{FGM}(u_1,u_2) = u_1u_2[1+\theta(1-u_1)(1-u_2)]$$

and

$$c_{FGM}(u_1,u_2) = [1+\theta(1-2u_1)(1-2u_2)]$$

Where $u_1 = F_1(x;\zeta_1), u_2 = F_2(y;\zeta_2)$ $\theta \in [-1.1]$ is a dependence parameter.

# 3. The Proposed Bivariate FGMBES-Exponential Distribution

In this section, we introduce the proposed bivariate distribution based on exponentiated sine–exponential marginals combined with the FGM copula.

## 3.1 Exponentiated Sine–Exponential Marginals

Let X be a random variable following the exponentiated sine exponential distribution. This distribution is obtained by applying the exponentiated sine transformation to the exponential baseline distribution, which enhances flexibility through an additional shape parameter.

Let G(x) and g(x) denote the CDF and PDF of the exponential distribution, respectively. The CDF and pdf of the ES–Exponential distribution is given by

$$F(x) = \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha} \quad , \qquad G(x) = 1 - e^{-\left(\frac{x}{\lambda_1}\right)}, \;\; \alpha > 0$$

$$f(x)=\frac{\alpha\pi}{2}g(x)\cos\left(\frac{\pi}{2}G(x)\right)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1},\mathrm{g(x)}=\frac{1}{\lambda}e^{-\left(\frac{x}{\lambda_1}\right)}$$

where $\alpha$ is a shape parameter controlling skewness and tail behavior. This distribution is capable of modeling various hazard rate shapes, including increasing, decreasing, bathtub-shaped, and unimodal forms, making it suitable for lifetime data analysis.

## 3.2 Joint CDF and PDF

Let $\mathrm{X}$ and $\mathrm{Y}$ be two random variables with ES–Exponential marginal distributions $\mathrm{F_1(x)}$ and $\mathrm{F_2(y)}$. Using the FGM copula, the joint CDF and PDF of the proposed bivariate FGMBESE distribution is defined as

$$F_{FGMBESE}(x,y)=\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\left[1+\theta\left(1-\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right] \qquad (1)$$

$$f_{FGMBESE}(x,y)=\frac{\alpha_1\alpha_2\pi^2}{4\lambda_1\lambda_2}e^{-\left(\frac{x}{\lambda_1}+\frac{y}{\lambda_2}\right)}\cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1-1}\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2-1}\left[1+\theta\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right] \qquad (2)$$

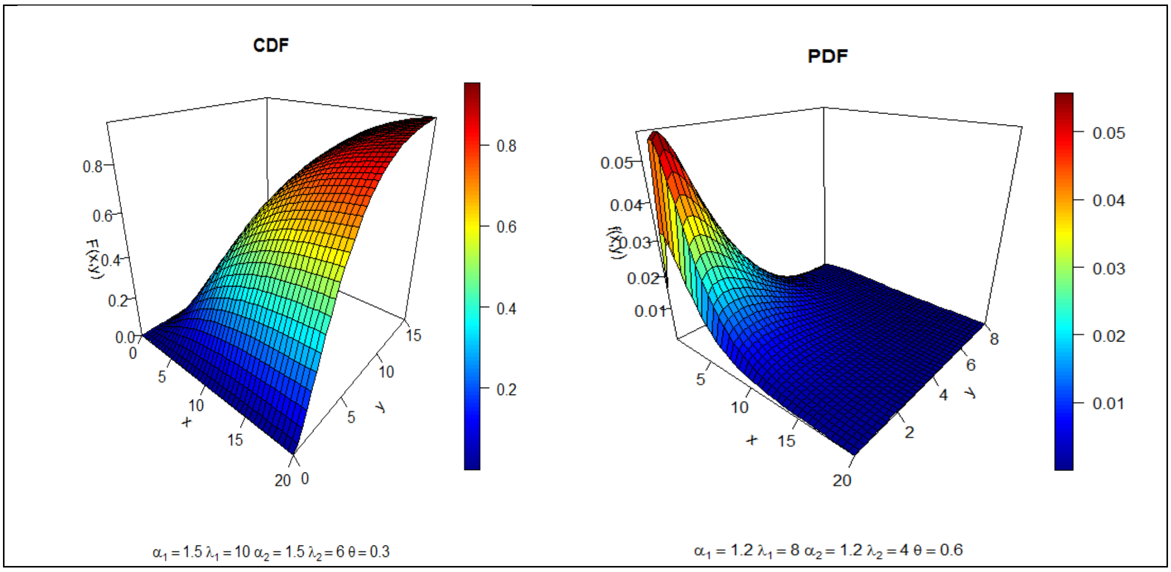


**Figure 1:** 3D plots of the joint PDF and cdf for the FGMBESE model

## 4. Statistical Properties

In this section, we discuss some statistical properties of the FGMBESE distribution such as marginal distributions, dependences measures, conditional distribution, reliability function, hazard rate function and moment generating function.

### 4.1 The Marginal and Conditional Distributions

The marginal density functions for $X\ and$ Y can be shown respectively as

$$f_1(x) = \frac{\alpha_1\pi}{2\lambda_1} e^{-\left(\frac{x}{\lambda_1}\right)} \cos\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1 - 1} \quad \lambda_1, \alpha_1 > 0$$

$$f_2(y) = \frac{\alpha_2\pi}{2\lambda_2} e^{-\left(\frac{y}{\lambda_2}\right)} \cos\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2 - 1} \quad \lambda_2, \alpha_2 > 0$$

The conditional probability distribution of $X$ given Y is given as

$$f_{FGMBESE}(x|y) = f_1(x)\left[1 + \theta\left(1 - 2F_1(x)\right)\left(1 - F_2(y)\right)\right] = \frac{\alpha_1\pi}{2\lambda_1} e^{-\left(\frac{x}{\lambda_1}\right)} \cos\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1 - 1}\left[1 + \theta\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right]$$

The conditional cdf of X $given$ Y is given as

$$F_{FGMBESE}(x|y) = F_1(x)\left[1 + \theta\left(1 - F_1(x)\right)\left(1 - 2F_2(y)\right)\right] = \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1} \times \left[1 + \theta\left(1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right]$$

### 4.2 Reliability and Hazard Rate Function

Following Sreelakshmi (2018), The relationship between copulas and reliability functions establishes that the bivariate survival function can be expressed as:

$$S_{FGMBESE}(x,y) = \left[1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right]\left[1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right] \times \left[1 + \theta\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right]$$

The conditional survival function of $X$ given Y is given as

$$S_{X|Y}(x|y) = 1 - F_X(x)\left[1 + \theta\left(1 - F_X(x)\right)\left(1 - 2F_Y(y)\right)\right] = 1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1} \times \left[1 + \theta\left(1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right]$$

The bivariate hazard (failure) rate function, introduced by Basu (1971), measures the instantaneous failure rate of both components given that both have survived until times x and y:

$$h_{FGMBESE}(x,y) = \frac{f_{FGMBESE}(x,y)}{S_{FGMBESE}(x,y)}$$

Where $f_{FGMBESE}(x,y), S_{FGMBESE}(x,y)$ in equations (2) , (3).

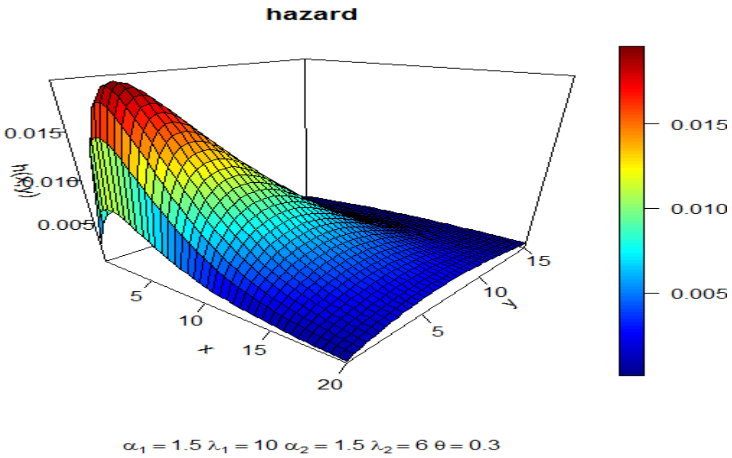


**Figure 2:** 3D plots of the joint hazard rate function for the FGMBESE

### 4.3 Moment Generating Function

Let $(X,Y)$ denote abivariate random variable with the probability density function of FGMBESE. Then, the moment generating function of $(X,Y)$ is given by

$$M_{X,Y}(t_1,t_2) = E\left[\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(t_1X)^{n_1}}{n_1!}\frac{(t_2Y)^{n_2}}{n_2!}\right] = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{{t_1}^{n_1}}{n_1!}\frac{{t_2}^{n_2}}{n_2!}E[X^{n_1}Y^{n_2}]$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{{t_1}^{n_1}}{n_1!}\frac{{t_2}^{n_2}}{n_2!}\acute{\mu}_{n_1,X}\acute{\mu}_{n_2,Y}\left[1+\theta(1-2\psi_{n_1}^{(X)})(1-2\psi_{n_2}^{(Y)})\right]$$

Where $\acute{\mu}_{n_1,X} = E[X^{n_1}] = \int_0^{\infty} x^{n_1} f_1(x)dx$ , $\acute{\mu}_{n_2,Y} = E[Y^{n_2}] = \int_0^{\infty} y^{n_2} f_2(y)dy$

and

$$\psi_{n_1}^{(X)} = \frac{E[X^{n_1}F_1(x)]}{E[X^{n_1}]} = \frac{\int_0^{\infty} x^{n_1} f_1(x)F_1(x)dx}{\int_0^{\infty} x^{n_1} f_1(x)dx}, \quad \psi_{n_2}^{(Y)} = \frac{E[Y^{n_2}F_2(y)]}{E[Y^{n_2}]} = \frac{\int_0^{\infty} y^{n_2} f_2(y)F_2(y)dy}{\int_0^{\infty} y^{n_2} f_2(y)dy}$$

.

## 5. Dependence Measures for FGM Copula

In this section, we introduce some measures of dependence based on copulas for the FGMBESE, such as Kendall's $\tau_c$, Blomqvist's medial correlation coefficient $b_c$ and Spearman's footrule coefficient $\delta_c$, which are defined by Nelsen (2006) and Popovic et al.(2020).

### 5.1 Kendall's Tau Correlation ( $\tau_c$)

The difference between the probabilities of concordance and discordance of $(X_1, X_2)$ and $(\widetilde{X_1}, \widetilde{X_2})$ is denoted as Kendall's tau $\tau_c$ , the Kendall's tau is defined as: $\tau_c = 4\int_0^1 \int_0^1 c(u_1,u_2)C(u_1,u_2)\, du_1 du_2 - 1$ .

For the FGM copula with parameter θ, Kendall's tau $\boldsymbol{\tau_{c_\theta}} = \frac{\mathbf{2\theta}}{\mathbf{\theta}}$, indicating a moderate level of dependence. Specifically, $\tau_{c_\theta}$ranges between $\frac{-2}{9}, \frac{2}{9}$

### 5.2 Blomqvist's medial correlation coefficient ($\mathbf{b_c}$)

Using population medians $x_0$ and $y_0$, Blomqvist (1950) proposed and studied a measure called the medial correlation coefficient, denoted as Blomqvist's medial correlation coefficient $(b_c)$. In copula terms, Blomqvist's medial correlation coefficient is defined as $b_c = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1$. For the FGM copula, Blomqvist's medial correlation coefficient is $\boldsymbol{b_{c_\theta}} = \frac{\boldsymbol{\theta}}{\mathbf{4}}$ and this means that $b_{c_\theta} \in [\frac{-1}{4}, \frac{1}{4}]$.

### 5.3 Spearman's footrule coefficient $\boldsymbol{\delta_c}$

Bekrizadeh (2021) and Bukovšek et al.(2021) defined Spearman's footrule coefficient for X and Y as $\delta_c = 6\int_0^1 C(u_1,u_2)du_1 - 2$. For the FGM copula, Spearman's footrule coefficient is $\boldsymbol{\delta_c} = \frac{\boldsymbol{\theta}}{\mathbf{5}}$ and this means that $\delta_{c_\theta} \in [\frac{-1}{5}, \frac{1}{5}]$.

## 6. Maximum Likelihood Estimation

Elaal and Jarwan (2017), discussed the maximum likelihood estimator to estimate all model parameters jointly, it is a one-step parametric method. Therefore, the log-likelihood is given as

$$\ell_{FGM}(\Theta) = \sum_{i=1}^{n} \log f_{FGMBESE}(x_i, y_i; \Theta)$$

Where $\Theta = (\alpha_1, \alpha_2, \zeta_1, \zeta_2, \theta)$ and $f_{FGMBESE}(x_i, y_i; \Theta)$ is given in Equation(2 ). Substituting the density function, we have

$$\ell_{\mathrm{FGM}}(\Theta) = \sum_{i=1}^{n} \Bigg\{ \log\left(\frac{\alpha_1\alpha_2\pi^2}{4\lambda_1\lambda_2} e^{-\left(\frac{x_{1i}}{\lambda_1}+\frac{y}{\lambda_2}\right)} \cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right) \cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right) + \log\left(\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1-1} \left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2-1}\right) + \log\left[1+\theta\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right]\Bigg\}.$$

For a sample of size n, the log- likelihood is given by

$$\ell_{\mathrm{FGM}}(\Theta) = n\log\left(\frac{\pi^2}{4}\right) + n\log(\alpha_1) + n\log(\alpha_2) - \sum_{i=1}^{n}\left(\frac{x_{1i}}{\lambda_1}+\frac{y_{2i}}{\lambda_2}\right) + \sum_{i=1}^{n}\log\cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right) + \sum_{i=1}^{n}\log\cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)$$
$$+(\alpha_1-1)\sum_{i=1}^{n}\log\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right] + (\alpha_2-1)\sum_{i=1}^{n}\log\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]$$
$$+\sum_{i=1}^{n}\log\left[1+\theta\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\right].$$

The maximum likelihood estimates are obtained by differentiating the above equation with respect to $\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta$ and equating the resulting expressions to zero.

$$\frac{\partial \ell_{\mathrm{FGM}}}{\partial \alpha_1} = 0\,,\quad \frac{\partial \ell_{\mathrm{FGM}}}{\partial \alpha_2} = 0,\quad \frac{\partial \ell_{\mathrm{FGM}}}{\partial \zeta_1} = 0,\quad \frac{\partial \ell_{\mathrm{FGM}}}{\partial \zeta_2} = 0,\quad \frac{\partial \ell_{\mathrm{FGM}}}{\partial \theta} = 0.$$

$$\frac{\partial \ell_{FGM}}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} lo\,g\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right] - 2\theta\sum_{i=1}^{n}\frac{\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1} log\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]\cdot\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}{1+\theta\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}$$

$$\frac{\partial \ell_{FGM}}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{i=1}^{n} log\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right] - 2\theta\sum_{i=1}^{n}\frac{\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2} log\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]}{1+\theta\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}$$

$$\frac{\partial \ell_{\mathrm{FGM}}}{\partial \lambda_1} = \frac{-n}{\lambda_1} + \sum_{i=1}^{n}\frac{x_{1i}}{{\lambda_1}^2} + \sum_{i=1}^{n}\tan\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\times\frac{\pi x_{1i}}{2{\lambda_1}^2}e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}$$
$$+\sum_{i=1}^{n}\frac{\pi x_{1i}\theta\alpha_1}{{\lambda_1}^2}\frac{\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1-1}\cos\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}}{1+\theta\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1-2\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}$$

$$\frac{\partial \ell_{\mathrm{FGM}}}{\partial \lambda_2} = \frac{-n}{\lambda_2} + \sum_{i=1}^{n} \frac{y_{2i}}{{\lambda_2}^2} + \sum_{i=1}^{n} \tan\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right) \times \frac{\pi y_{2i}}{2{\lambda_2}^2} e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}$$

$$+ \sum_{i=1}^{n} \frac{\pi y_{2i} \theta \alpha_2}{{\lambda_2}^2} \frac{\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2 - 1} \cos\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right) e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}}{1 + \theta\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}$$

$$\frac{\partial \ell_{\mathrm{FGM}}}{\partial \theta} = \sum_{i=1}^{n} \frac{\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}{1 + \theta\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{x_{1i}}{\lambda_1}\right)}\right)\right)\right]^{\alpha_1}\right)\left(1 - 2\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\left(\frac{y_{2i}}{\lambda_2}\right)}\right)\right)\right]^{\alpha_2}\right)}$$

we obtain the non-linear normal equations. So, the solution has to be obtained numerically using optimization methods such as Newton-Raphson, gradient descent, or BFGS algorithm.

## 7. Real Data Application

This section evaluates the performance of the proposed FGMBESE distribution in modeling real-life data through a comprehensive comparison with established bivariate models. The models compared include a bivariate Kumaraswamy Type Exponential (BKE) which was studied by Mirhosseini et al.(2016) , FGM bivariate Weibull (FGMBW) which was discussed by Almetwally et al.(2020), bivariate FGM power Lomax (BFGMPLx), for Qura et al.(2023), FGM bivariate generalized exponential (FGMBGE) which was mentioned by by Abd Elaal and Jarwan (2017).

Goodness-of-fit was assessed using Akaike’s information criterion (AIC), consistent AIC (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC).

We applied the models to a bivariate environmental dataset from Thomas and Jose (2020), containing average precipitation (mm) and average maximum temperature (°C) for 51 major U.S. cities (source: National Climatic Data Center, https://www.ncei.noaa.gov/access).

The results, summarized in Table 2, show that the FGMBESE model achieves the lowest values across all four criteria, confirming its superior ability to capture the dependence between precipitation and temperature.

**Table(1): Bivariate Model Estimation via MLE for environmental data**

| | Par. | $\alpha_1$ | $\lambda_1$ | $\alpha_2$ | $\lambda_2$ | $\theta$ |
|---|---|---|---|---|---|---|
| **FGMBESE** | **Estimate** | 4.749 | 64.422 | 2.151 | 12.364 | -0.239 |
| | **SE** | 1.150 | 7.816 | 0.449 | 1.784 | 0.459 |

**Table(2):Goodness of fit criteria and other models of environmental data**

| | **FGMBESE** | FGMBGE | BFGMPLx | BKE | FGMBW |
|---|---|---|---|---|---|
| -LL | **414.39** | 414.75 | 424.60 | 441.99 | 461.35 |
| AIC | **838.77** | 839.50 | 863.19 | 889.98 | 932.70 |
| BIC | **848.43** | 849.16 | 876.72 | 895.49 | 942.36 |
| CAIC | **840.11** | 840.38 | 865.80 | 890.49 | 934.03 |
| HQIC | **842.46** | 843.19 | 868.36 | 892.19 | 936.39 |

## 8. Conclusion

This paper has introduced and studied the FGM Bivariate Exponentiated Sine Exponential (FGMBESE) distribution. The investigation encompassed its key statistical and reliability properties, dependence measures, and a procedure for parameter estimation via the maximum likelihood method. The primary contribution of this work is demonstrated through an empirical application, where the FGMBESE model is shown to provide a superior fit to real lifetime data. Compared to established bivariate models, the proposed distribution achieves this enhanced fit without introducing excessive complexity through additional parameters, making it a parsimonious and effective tool for modeling environmental lifetime data.

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